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THE COHOMOLOGY OF GROUPS ACTING ON TREES

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1. Introduction

Suppose that a group G acts without inversion upon a tree X ([8], I-37). We shall write $V(X)$ for the set of vertices of X, $E(X)$ for the set of edges, G_P (resp. G_x) for the stabilizer of a vertex P (resp. edge x) and cd G for the cohomological dimension of G. Our purpose is to describe the cohomology of G in terms of the cohomology of the stabilizers of the vertices and edges of X, and our main result (Theorem 2) is that if

$$n_V = \sup\{\text{cd}\, G_P \,|\, P \in V(X)\},\ n_E = \sup\{\text{cd}\, G_x \,|\, x \in E(X)\}\ ,$$

then

$$n_V \leqslant \text{cd}\, G \leqslant \sup(2, n_V) \quad \text{if } n_E < n_V, \qquad \text{and}$$

$$n_V \leqslant \text{cd}\, G \leqslant n_V + 1 \quad \text{if } n_E = n_V\ .$$

Our constant reference for the structure of groups acting on trees will be Serre's notes [8]. We shall consider two types of special cases, or applications, of our results. For the first type we shall restrict our attention to the case where the graph of orbits $G\backslash X$ is also a tree. It turns out (loc. cit.) that this is the same as saying that G is a colimit of a tree of groups, and what we have in this case is a description of the cohomology of colimits of trees of groups (Corollaries 1.1 and 1.2). The second application deals with one-relator groups and their generalizations. Every group G with one defining relator r of length $\geqslant 2$ can be made to act upon a tree X in such a way that the stabilizers of its vertices are embeddable in another one-relator group G_0, whose defining relator has length less than that of r. If r is a proper power, cd $G = \infty$ trivially; if not then Lyndon's inequality [7] cd $G \leqslant 2$ becomes a corollary of our main theorem. Finally, we remark that the techniques of this paper also lead to a generalization of Lyndon's theorem contained in [4].

2. Suppose that a group G acts without inversion upon a tree X, and let T be a maxi-

mal tree in the graph of orbits $Y = G\backslash X$ ([8], I-37, Prop. 14). We know by I-84, Theorem 13, that G is isomorphic to the fundamental group $\pi_1(G, Y, T)$ of a graph of groups (G, Y), defined as on p. I-83, the group at every vertex (resp. edge) of Y being a stabilizer of a vertex (resp. edge) of X. Choose an orientation of Y, and let $E(Y)^+$ denote the set of its positive edges. Write $S = E(Y)^+ - E(Y)^+ \cap E(T)$; let $F(S)$ denote the free group on the set S and for every positive edge y of Y, let

$$\langle y \rangle = \begin{cases} 1 & \text{if } y \in E(T) \\ \text{the image of } y \text{ in } F(S) & \text{if not.} \end{cases}$$

The *universal cover* $\tilde{Y}$ of Y w.r.t. T is the tree defined by:

$$E(\tilde{Y}) = F(S) \times E(Y), \qquad V(\tilde{Y}) = F(S) \times V(Y),$$

the initial point $\tilde{d}_0(f, y)$ and terminal point $\tilde{d}_1(f, y)$ of an edge (f, y) of $\tilde{Y}$ being the following:

$$\tilde{d}_0(f, y) = \begin{cases} (f, d_0(y)) & \text{if } y \in E(Y)^+ \\ (f\langle y \rangle^{-1}, d_0(y)) & \text{if not} \end{cases}$$

$$\tilde{d}_1(f, y) = \begin{cases} (f\langle y \rangle, d_1(y)) & \text{if } y \in E(Y)^+ \\ (f, d_1(y)) & \text{if not,} \end{cases}$$

where $d_0(y)$ and $d_1(y)$ denote the initial point and end point respectively of the edge y in Y. The free group $F(S)$ acts freely on $\tilde{Y}$ (i.e. all stabilizers are trivial):

$$f(f', P) = (ff', P), \quad f(f', y) = (ff', y), \quad f, f' \in F(S), \quad P \in V(Y), \quad y \in E(Y).$$

As in the exercise on p. I-69 of Serre's notes, we shall define a graph of groups $(G, \tilde{Y})$, so that if we let $F(S)$ act on its colimit $K = \varinjlim (G, \tilde{Y})$ through the action of $F(S)$ on $\tilde{Y}$, the semi-direct product of K and $F(S)$ is canonically isomorphic to $\pi_1(G, Y, T)$. Note that $F(S)$ is the fundamental group $\pi_1(Y, T)$ of Y relative to T (I-82, ex. 1). To define $(G, \tilde{Y})$, we let

$$G_{(f,P)} = G_P, \quad G_{(f,y)} = G_y \quad \text{for all } f \in F(S), \quad P \in V(Y), \quad y \in E(Y);$$

and, if an edge y of Y has initial point P and end point Q in Y, we let

$$G_{(f,y)} \xhookrightarrow{\overline{(f,y)}} G_{\tilde{d}_0(f,y)} = G_P \quad \text{and} \quad G_{(f,y)} \xhookrightarrow{(f,y)} G_{\tilde{d}_1(f,y)} = G_Q$$

be the maps $G_y \xhookrightarrow{\bar{y}} G_p$ and $G_y \xhookrightarrow{y} G_Q$ associated with (G, Y). Let us denote the canonical maps: $G_{(f,P)} \to K$ by $m_{(f,P)}$. The action

$$\theta : F(S) \to \operatorname{aut}(K)$$

is then explicitly described by

$$\theta(f) \circ r_{(f',P)} = m_{(ff',P)} : G_{(f',P)} = G_{(ff',P)} \to K\,.$$

Let $K \times_\theta F(S)$ denote the semi-direct product of K and $F(S)$ w.r.t. this action; for every $y \in S$ let γ_y denote conjugation by the image $\beta(\langle y\rangle)$ of $\langle y\rangle$ in $K \times_\theta F(S)$:

$$\gamma_y : K \times_\theta F(S) \to K \times_\theta F(S)\,, \qquad \beta : F(S) \to K \times_\theta F(S)\,.$$

$$\gamma_y : z \to \beta\langle y\rangle z \beta(\langle y\rangle)^{-1} = \theta(\langle y\rangle)(z)$$

and for every $P \in V(Y)$, let α_P be the composition:

$$G_P = G_{(1,P)} \xhookrightarrow{m_{(1,P)}} K \to K \times_\theta F(S)\,.$$

For each $y \in E(Y)$ with $d_0(y) = P$ and $d_1(y) = Q$, one then has commutative diagrams:

$$\begin{tikzcd}
 & G_{\tilde{d}_0(1,y)} \arrow[d, equal] & \\
 & G_P \arrow[dr, "m_{\tilde{d}_0(1,y)}"] & \\
G_y \arrow[ur, hook, "\bar{y}"] \arrow[dr, hook, "y"'] & & K \\
 & G_Q \arrow[ur, hook, "m_{\tilde{d}_1(1,y)}"'] \arrow[d, equal] & \\
 & G_{\tilde{d}_1(1,y)} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G_P \arrow[r, hook, "\alpha_P"] & K \times_\theta F(S) \\
G_y \arrow[ur, hook, "\bar{y}"] \arrow[dr, hook, "y"'] & & \\
 & G_Q \arrow[r, hook, "\alpha_Q"] & K \times_\theta F(S) \arrow[uu, "\gamma_y"']
\end{tikzcd}$$

and one easily verifies that the maps $\beta : F(S) \to K \times_\theta F(S)$ and α_P ($P \in V(Y)$) are universal w.r.t. the property of rendering the second diagram commutative, in the sense that if $\beta' : F(S) \to L$, $\alpha'_P : G_P \to L$ ($P \in V(Y)$) are group homomorphisms with the same property, then there exists a unique map $\psi : K \times_\theta F(S) \to L$ such that $\psi \circ \beta = \beta'$ and $\psi \circ \alpha_P = \alpha'_P$ for all $P \in V(Y)$. But, this universal property characterizes $\pi_1(G, Y, T)$ (I-65,b), hence

$$K \times_\theta F(S) \simeq \pi_1(G, Y, T) \simeq G\,.$$

Henceforth, we shall identify these groups.

We shall view K, $G_{(f,P)}$, $((f,P) \in V(\tilde{Y}))$ and $G_{(f,y)}$, $((f,y) \in E(\tilde{Y}))$ as groups over K, i.e. as objects of the category (Grp, K), in the obvious way, so that K is $\varinjlim(G, \tilde{Y})$ also *in this category*. It is well-known [1, p. 255] that the category $Ab(Grp, K)$ of Abelian group objects in (Grp, K) is equivalent to the category K-*mod* of K-modules. The augmentation ideal of the group ring $\mathbf{Z}L$ of a group L will be denoted by I_L, and we now define K-modules:

$$\tilde{K} = I_K\,, \quad \tilde{G}_{(f,P)} = \tilde{G}_P = \mathbf{Z}K \otimes_{G_P} I_{G_P}\,, \quad (f,P) \in V(\tilde{Y})$$

$$\tilde{G}_{(f,y)} = \tilde{G}_y = \mathbf{Z}K \otimes_{G_y} I_{G_y}\,, \quad (f,y) \in E(\tilde{Y})\,.$$

It is also well-known [1, p. 255] that the functor

$$\mathrm{Diff}_K : (W \to K) \mapsto \widetilde{W} := ZK \otimes_W I_W$$

is the left adjoint of the embedding:

$$K\text{-}mod \approx Ab(Grp, K) \subset (Grp, K)\ ,$$

and $\widetilde{K}$ is therefore the colimit of the graph $(\widetilde{G}, \widetilde{Y})$ of K-modules. By this we mean that the maps $m_{(f,P)}$ $(f, P) \in V(\widetilde{Y})$), induce maps $\widetilde{m}_{(f,P)}: \widetilde{G}_{(f,P)} \to \widetilde{K}$,

$$\begin{tikzcd}
 & \widetilde{G}_{\widetilde{d}_0(f,y)} \arrow[dr, "\widetilde{m}_{\widetilde{d}_0(f,y)}"] & \\
\widetilde{G}_{(f,y)} \arrow[ur, "\tilde{\tilde{y}}"] \arrow[dr, "\tilde{y}"'] & & \widetilde{K} \\
 & \widetilde{G}_{\widetilde{d}_1(f,y)} \arrow[ur, "\widetilde{m}_{\widetilde{d}_1(f,y)}"'] &
\end{tikzcd}$$

commutes for all $(f, y) \in E(\widetilde{Y})$, and the maps $\widetilde{m}_{(f,P)}$ are universal w.r.t. this property. Let us define:

$$\mu: \bigoplus_{(f,y)\in E(\widetilde{Y})^+} \widetilde{G}_{(f,y)} \longrightarrow \bigoplus_{(f,P)\in V(\widetilde{Y})} \widetilde{G}_{(f,P)}$$

by

$$\mu | \widetilde{G}_{(f,y)} = \lambda_{\widetilde{d}_0(f,y)} \cdot \tilde{\tilde{y}} - \lambda_{\widetilde{d}_1(f,y)} \cdot \tilde{y}\ ,$$

where

$$\lambda_{(f,P)} : \widetilde{G}_{(f,P)} \hookrightarrow \bigoplus_{(f,P)\in V(\widetilde{Y})} \widetilde{G}_{(f,P)}$$

denotes the canonical map. The map μ is injective, because $\widetilde{Y}$ is a tree ([8], I-69). The maps

$$\widetilde{m}_{(f,P)} : \widetilde{G}_{(f,P)} \to \widetilde{K}$$

induce

$$\varphi: \bigoplus_{(f,P)\in V(\widetilde{Y})} \widetilde{G}_{(f,P)} \to \widetilde{K}\ ,$$

which is onto, since K is generated by the images $m_{(f,P)}(G_{(f,P)})$ of $G_{(f,P)}$. Now, every map

$$\nu: \bigoplus_{(f,P)\in V(\widetilde{Y})} \widetilde{G}_{(f,P)} \to B$$

in K-mod satisfies $\nu \cdot \mu = 0$ iff the maps

$$\nu_{(f,P)} = \nu \circ \lambda_{(f,P)}: \widetilde{G}_{(f,P)} \to B$$

satisfy

$$\nu_{\widetilde{d}_0(f,y)} \cdot \tilde{\tilde{y}} = \nu_{\widetilde{d}_1(f,y)} \cdot \tilde{y} \quad \text{for all } (f,y) \in E(\widetilde{Y})^+\ .$$

It follows that φ is the cokernel of μ, and we have proved:

Proposition.

$$0 \to \bigoplus_{(f,y)\in E(\widetilde{Y})^+} \widetilde{G}_{(f,y)} \xrightarrow{\cdot\mu} \bigoplus_{(f,P)\in V(\widetilde{Y})} \widetilde{G}_{(f,P)} \xrightarrow{\varphi} \widetilde{K} \to 0$$

is an exact sequence of K-modules.

Suppose now that A is a G-module. The group $F(S)$ acts upon the Abelian group $\mathrm{Der}(K,A)$ of derivation from K into A by the usual formula:

$$(f\delta)(k) = f\,\delta(f^{-1}kf) \quad (f\in F(S), \quad \delta\in\mathrm{Der}(K,A), \quad k\in K)\,,$$

and this induces an $F(S)$-module action on the value at A of the qth right derived functor $H^{q+1}(K,\text{-})$ of $\mathrm{Der}(K,\text{-})$, ($q \geqslant 1$). We point out, for future reference, that this action is the right one for the spectral sequence associated with: $1\to K\to G\to F(S)\to 1$ [6, p. 303]. Note that we identify $G_{(f,P)}$ and $G_{(1,P)}$; but *not* their actions (via $m_{(f,P)}$ and $m_{(1,P)}$ resp.) on A.

The following theorem will be proved by applying the functor $\mathrm{Hom}_K(\text{-},A)$ to the short exact sequence of the Proposition.

Theorem 1. *For every Abelian group B, we let B^* denote the co-induced $F(S)$-module* $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}F(S),B)$. *To every G-module A there corresponds an exact sequence of $F(S)$-modules*:

$$0\to \mathrm{Der}(K,A)\to \Big(\prod_{P\in V(Y)} \mathrm{Der}(G_P,A)\Big)^* \to \Big(\prod_{y\in E(Y)^+} \mathrm{Der}(G_y,A)\Big)^*$$

$$\to H^2(K,A)\to \Big(\prod_{P\in V(Y)} H^2(G_P,A)\Big)^* \to \Big(\prod_{y\in E(Y)^+} H^2(G_y,A)\Big)^* \to H^3(K,A)\to \dots$$

Corollary 1.1. *If Y is a tree, the above long exact sequence becomes the following exact sequence of Abelian groups*:

$$0\to \mathrm{Der}(G,A)\to \prod_{P\in V(Y)} \mathrm{Der}(G_P,A)\to \prod_{y\in E(Y)^+} \mathrm{Der}(G_y,A)$$

$$\to H^2(G,A)\to \prod_{P\in V(Y)} H^2(G_P,A)\to \prod_{y\in E(Y)^+} H^2(G_y,A)\to H^3(G,A)\to \dots$$

Proof. In this case $S=\emptyset$ and $G=\pi_1(G,Y,T)=\pi_1(G,T,T)=\varinjlim(G,T)=K$ ([8], I-66, Ex. 2).

Corollary 1.2. *Given a tree of groups (G,T), with colimit G_T. Then*

$$\mathrm{cd}\,G_T \leqslant \sup\,\{\mathrm{cd}\,G_P \mid P\in V(T)\} + 1\,.$$

Proof. Corollary 1.1 and [8], I-61, Theorem 10.

Proof of Theorem 1. We first take note of the following natural isomorphisms of functors from K-*mod* into the category of Abelian groups:

$$\mathrm{Hom}_K(I_K, -) \simeq \mathrm{Der}_K(K, -)$$

$$\mathrm{Hom}_K\Big(\bigoplus_{(f,P)\in V(\tilde{Y})} \tilde{G}_{(f,P)}, -\Big) \simeq \prod_{(f,P)\in V(\tilde{Y})} \mathrm{Der}(G_{(f,P)}, -)$$

$$\mathrm{Hom}_K\Big(\bigoplus_{(f,y)\in E(\tilde{Y})^+} \tilde{G}_{(f,y)}, -\Big) \simeq \prod_{(f,y)\in E(\tilde{Y})^+} \mathrm{Der}(G_{(f,y)}, -)\,.$$

Next, we note that every K-injective module is also $G_{(f,P)}$-injective and $G_{(f,y)}$-injective, and products are of course exact functors. It follows that the short exact sequence of the Proposition gives rise to a sequence of functors and natural transformations:

$$\Sigma : 0 \to \mathrm{Der}(K,-) \to \prod_{(f,P)\in V(\tilde{Y})} \mathrm{Der}(G_{(f,P)}, -) \to \prod_{(f,y)\in E(\tilde{Y})^+} \mathrm{Der}(G_{(f,y)}, -) \to 0\,,$$

which is exact on K-injectives, so that one has a long exact sequence ([6], Theorem 6.3, p. 137) of Abelian groups

$$0 \to \mathrm{Der}(K,C) \to \prod_{(f,P)\in V(\tilde{Y})} \mathrm{Der}(G_{(f,P)}, C) \to \prod_{(f,P)\in E(\tilde{Y})^+} \mathrm{Der}(G_{(f,y)}, C)$$

$$\to H^2(K,C) \to \prod_{(f,P)\in V(\tilde{Y})} H^2(G_{(f,P)}, C) \to \prod_{(f,y)\in E(\tilde{Y})^+} H^2(G_{(f,y)}, C) \to H^3(K,C) \to \ldots$$

for every K-module C. We now replace C by a G-module A. For fixed $P \in V(Y)$, the groups $G_{(f,P)}$ are all equal to G_P, and the groups $G_{(f,y)}$ are all equal to G_y. The Abelian groups

$$\prod_{(f,P)\in V(\tilde{Y})} \mathrm{Der}(G_{(f,P)}, A)\,, \qquad \prod_{(f,P)\in E(\tilde{Y})^+} \mathrm{Der}(G_{(f,y)}, A)$$

are therefore isomorphic to the underlying Abelian groups of the co-induced modules.

$$\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}F(S), (\prod_{P\in V(Y)} \mathrm{Der}(G_P, A)))\,, \qquad \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}F(S), (\prod_{y\in E(Y)^+} \mathrm{Der}(G_y, A)))$$

respectively. The same applies to the products of the cohomology groups that appear in the long exact sequence. In order for the natural transformations in the sequence Σ, when evaluated at A, to be homomorphisms of $F(S)$-modules, we want the $F(S)$-module structures on the last two terms to be determined by commutative squares of the form:

$$\begin{array}{ccc}
\prod_{(f,P)\in V(\tilde{Y})} \mathrm{Der}(G_{(f,P)},A) & \xrightarrow{f'} & \prod_{(f,P)\in V(\tilde{Y})} \mathrm{Der}(G_{(f,P)},A) \\
\Big\downarrow{\scriptstyle \pi(f,P)} & & \Big\downarrow{\scriptstyle \pi(f'f,P)} \\
\mathrm{Der}(G_{(f,P)},A) & \xrightarrow{\sigma_{f'}} & \mathrm{Der}(G_{(f'f,P)},A)
\end{array}$$

where f' denotes scalar multiplication by f', the vertical maps are canonical projections and $\sigma_{f'}$ is *well-defined* on

$$\mathrm{Der}(G_P,A) = \mathrm{Der}(G_{(f,P)},A) = \mathrm{Der}(G_{(f'f,P)},A)$$

by $\sigma_{f'}(\delta)(g) = f'\delta(g)$, $g \in G_P$. This structure agrees, via the isomorphism

$$\sigma : \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}F(S), (\prod_{P\in V(Y)} \mathrm{Der}(G_P,A))) \to \prod_{(f,P)\in V(\tilde{Y})} \mathrm{Der}(G_{(f,P)},A)$$

defined by:

$$(\pi_{(f,P)} \circ \sigma)(\varphi)(g) = f(\pi_{(1,P)}(\varphi(f^{-1}))(g)),$$

with the co-induced structure: $(f'\varphi)(f) = \varphi(ff')$. One has a similar commutative square for the other product in the sequence Σ. By naturality in A of these commutative squares, one also has similar commutative squares for the derived functors of the products in the sequence Σ, and this implies that our long exact sequence is indeed an exact sequence in the *category of $F(S)$-modules.*

Theorem 2. *Let G act on a tree X, and let*

$$n_E = \sup\{\mathrm{cd}\, G_x \mid x \in E(X)\} \leqslant \infty$$

$$n_V = \sup\{\mathrm{cd}\, G_P \mid P \in V(X)\} \leqslant \infty .$$

Then

$$n_V \leqslant \mathrm{cd}\, G \leqslant \sup(2, n_V) \quad \text{if } n_E < n_V , \qquad \text{and}$$

$$n_V \leqslant \mathrm{cd}\, G \leqslant n_V + 1 \quad \text{if } n_E = n_V .$$

Proof. We may assume without loss of generality that $n_V < \infty$. Let $m = \sup(2, n_V)$. If $m = 2$, the groups $H^{m-1}(G_y, A)$ should be replaced, in what follows, by $\mathrm{Der}(G_y, A)$. The long exact sequence of Theorem 1 contains:

$$(\prod_{y\in E(Y)^+} H^{m-1}(G_y,A))^* \xrightarrow{\omega_1} H^m(K,A)$$

$$\to (\prod_{P\in V(Y)} H^m(G_P,A))^* \xrightarrow{\omega_2} (\prod_{y\in E(Y)^+} H^m(G_y,A))^* \to H^{m+1}(K,A) \to 0 .$$

If $n_E < n_V$, we can extract from this two short exact sequences of $F(S)$-modules:

$$0 \to \ker \omega_1 \to \Big(\prod_{y\in E(Y)^+} H^{m-1}(G_y, A)\Big)^* \to \operatorname{Im} \omega_1 \to 0$$

$$0 \to \operatorname{Im} \omega_1 \to H^m(K, A) \to \Big(\prod_{P\in V(Y)} H^m(G_P, A)\Big)^* \to 0$$

each of which contains a co-induced $F(S)$-module. The associated long exact sequences contain:

$$0 = H^1\Big(F(S), \big(\prod_{y\in E(Y)^+} H^{m-1}(G_y, A)\big)^*\Big) \to H^1(F(S), \operatorname{Im} \omega_1) \to H^2(F(S), \ker \omega_1) = 0$$

$$0 = H^1(F(S), \operatorname{Im} \omega_1) \to H^1(F(S), H^m(K, A)) \to H^1\Big(F(S), \big(\prod_{P\in V(Y)} H^m(G_P, A)\big)^*\Big) = 0.$$

So, $H^1(F(S), H^m(K, A)) = 0$ and also $H^q(K, A) = 0$ for $q > m$. Therefore, by the Lyndon–Hochschild–Serre spectral sequence

$$H^q(G, A) = 0 \qquad \text{for all } q \geqslant m + 1 .$$

If $n_E = n_V$, we apply $H(F(S), \text{-})$ to the short exact sequence:

$$0 \to \operatorname{Im} \omega_2 \to \Big(\prod_{y\in E(Y)^+} H^m(G_y, A)\Big)^* \to H^{m+1}(K, A) \to 0$$

to obtain $H^1(F(S), H^{m+1}(K, A)) = 0$. Also, $H^q(K, A) = 0$ for $q > m+1$, and the spectral sequence now gives: $\operatorname{cd} G \leqslant m + 1$.

3. One-relator groups

We shall assume that the element r, defining relator for the one-relator group $G = F/(r)$, is not a proper power in the free group F, and our purpose is to deduce from Theorem 2, by induction on the word length of r, that $\operatorname{cd} G \leqslant 2$. We may assume without loss of generality that F is finitely generated: $F = F(x_0, x_1, \ldots, x_n)$ and r is a cyclically reduced word in which each of the letters $x_0, x_1, \ldots, x_n$ appears $(n \geqslant 0)$.

We now claim that we may also suppose that the exponent sum of one of the generators in r is zero. First, if only one generator appears in r, there is nothing to prove, and if x_0 (resp. x_1) appears with non-zero exponent sum α (resp. β), then we may adjoin a βth root d of x_0 to $G = F/(r)$ and put $e = x_1 d^\alpha$ (as already outlined in [2, p. 172]) in order to obtain a push-out diagram in the category of groups:

$$\begin{tikzcd} d^{\beta} & F(d) \arrow[r] & \tilde{G} = F(d,e,x_2,\dots,x_n)/(\tilde{r}) \\ x_0 \arrow[u, leftrightarrow] & F(x_0) \arrow[u, hook] \arrow[r, hook] & G \arrow[u, hook] \end{tikzcd}$$

where $\tilde{r}$ is the word

$$\tilde{r}(d,e,x_1,\dots,x_n) = r(d^{\beta}, ed^{-\alpha}, x_2, \dots, x_n)$$

obtained from r by substituting d^{β} for x_0 and $ed^{-\alpha}$ for x_1 in r. The Mayer–Vietoris sequence associated with this push-out [6, p. 221] gives $\mathrm{cd}\, G \leqslant 2$ iff $\mathrm{cd}\, \tilde{G} \leqslant 2$. Since $\tilde{r}$ has exponent sum zero in the generator d, it belongs to the normal subgroup $H = (e, x_2, \dots, x_n)$ of $F(d,e,x_2,\dots,x_n)$ generated by $e, x_2, \dots, x_n$. We note that H is freely generated by the conjugates d^jed^{-j}, $d^jx_id^{-j}$ ($j \in \mathbf{Z}$, $i = 2,\dots,n$). Moreover, $\tilde{r}$ has shorter length, when expressed as a work in these generators, than the original relator r. The same shortening would occur if x_0 already had exponent sum zero in r, and r is expressed as a word in the generators

$$x_{ij} = x_0{}^j x_i x_0{}^{-j} \qquad (i = 1,\dots,n;\ \ j \in \mathbf{Z})$$

for the free normal subgroup $(x_1,\dots,x_n)$ of F. We may therefore revert to the original notation and write x_0 and x_1 instead of d and e, so that

$$H = (x_1,\dots,x_n) = F(x_{i,j} : i = 1,\dots,n;\ j \in \mathbf{Z}) .$$

We also write $r_j = x_0{}^j r x_0{}^{-j}$ ($j \in \mathbf{Z}$).

We suppose that r_0 (a finite word) belongs to the subgroup

$$F_0 = F(x_{1,m_1}, x_{1,m_1+1}, \dots, x_{1,M_1}, x_{2,m_2}, \dots, x_{2,M_2}, \dots, x_{n,m_n}, \dots, x_{n,M_n})$$

of H, and adopt the convention that $m_i = M_i = 0$ if there is no $j \in \mathbf{Z}$ such that x_{ij} appears in r. Let R_j be the normal subgroup of

$$F_j = F(x_{1,m_1+j}, \dots, x_{1,M_1+j}, x_{2,m_2+j}, \dots, x_{2,M_2+j}, \dots, x_{n,m_n+j}, \dots, x_{n,M_n+j})$$

generated by r_j; put $G_j = F_j/R_j$ and

$$V_j = F(x_{1,m_1+j+1}, \dots, x_{1,M_1+j}, x_{2,m_2+j+1}, \dots, x_{2,M_2+j}, \dots, x_{n,m_n+j+1}, \dots, x_{n,M_n+j}).$$

In the following (infinite) diagram, α_j maps each $x_{i,h}$ that appears as a generator for V_j onto its natural image in G_j, and α_j' maps $x_{i,h}$ onto its natural image in G_{j+1}:

$$\begin{tikzcd}[column sep=small] & V_{-2} \arrow[dl, "\alpha_{-2}"'] \arrow[dr, "\alpha'_{-2}"] & & V_{-1} \arrow[dl, "\alpha_{-1}"'] \arrow[dr, "\alpha'_{-1}"] & & V_0 \arrow[dl, "\alpha_0"'] \arrow[dr, "\alpha'_0"] & & V_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha'_1"] & & V_2 \arrow[dl, "\alpha_2"'] \arrow[dr, "\alpha'_2"] & \cdots \\ \cdots\ G_{-2} & & G_{-1} & & G_0 & & G_1 & & G_2 & & G_3\ \cdots \end{tikzcd}$$

It follows from Magnus' Freiheitsatz [7] that the maps α_j and α_j' are injective. The normal subgroup $R = (r)$ of F generated by r is the normal subgroup of H generated by $\{r_j | j \in \mathbf{Z}\}$ and the group $K = H/R$ is the colimit (or generalized push-out) of the above diagram. In the terminology of [8], $K = \varinjlim (G, T)$, where T is the tree $\Gamma(F(x_0), \{x_0\})$ and (G, T) is the graph of groups

$$\cdots\; G_{-2} \xrightarrow{V_{-2}} G_{-1} \xrightarrow{V_{-1}} G_0 \xrightarrow{V_0} G_1 \xrightarrow{V_1} G_2 \;\cdots$$

The free group $F(x_0)$ acts upon the above diagram and upon K in the obvious way, and G is the semi-direct product of K and $F(x_0)$. We may also describe G as the fundamental group $\pi_1(G, Y, T')$, where Y is the graph with one vertex P and one edge, $T' = \{P\}$ and the group $G_P = G_0$, associated with P, may be described as the stabilizer of the image of P in the universal cover $\widetilde{X} = \widetilde{X}(G, Y, T')$ of (G, Y, T') (see [8], I-66, I-78 and I-84, Theorem 13). The stabilizers of all the vertices of the tree $\widetilde{X}$ are isomorphic to G_0, and the stabilizers of all the edges are isomorphic to the free group V_0 (I-78 and I-80, Theorem 12). The defining relator r_0 of G_0 has length less than that of r, and we may assume, by induction, that cd $G_0 \leqslant 2$. So as a consequence of Theorem 2, we have

Theorem 3 (Lyndon). *The cohomological dimension of a one-relator group is* $\leqslant 2$ *if the defining relator is not a proper power in the free group; otherwise it is* ∞.

Remark. Let P be the free product of finitely generated torsion-free Abelian groups and let (r) be the normal subgroup of P generated by some element r of P. In [4], the author has described the cohomology of $P/(r)$ by techniques similar to the ones above; thus generalizing Lyndon's Theorem. It is not difficult to see that the work done there also fits into the above general framework of the cohomology of groups acting on trees.

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